(DM 23)

M.Sc. (Final) Mathematics DEGREE EXAMINATION, MAY 2007

Paper III - **ANALYTICAL NUMBER THEORY AND GRAPH THEORY**

Time : Three hours Maximum : 100 marks

Answer any FIVE out of the given TEN questions
selecting atleast TWO from each Part.

1. (a) State and prove Euler's summation formula.

   (b) If $x \geq 1$, show that $\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right)$ where C is Euler's constant.

2. (a) If $x \geq 1$ and $\alpha > 0$, $\alpha \neq 1$, show that $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + D\left|x^\beta\right|$, where $\beta = \max\{1, \alpha\}$.

   (b) Show that the set of lattice points visible from the origin has density $6/\pi^2$.

3. (a) For x > 0, show that $0 \leq \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}$.

   (b) For $x \geq 2$, show that $\theta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} dt$.

4. (a) For every integer $n \geq 2$, show that $\frac{n}{6\log n} < \pi(n) < 6\frac{n}{\log n}$.

PART B

5. (a) Show that a graph G is disconnected if and only if its vertex set V can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

   (b) If a graph has exactly two vertices of odd degree, show that there must be a path joining these two vertices.

6. (a) In a connected graph G with exactly 2k odd vertices, show that there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a universal graph.

   (b) Show that a connected graph G is an Euler graph if and only if it can be decomposed into circuits.

7. (a) If in a graph *G* there is one and only one path between every pair of vertices, show that *G* is a tree.

   (b) Show that a graph *G* with *n* vertices, *n-1* edges and no circuits is connected.

8. (a) Show that energy circuit has an even number of edges in common with any cut-set.

   (b) Show that the vertex connectivity of any graph *G* can never exceed the edge connectivity of *G*.

9. (a) Show that a connected planar graph with *n* vertices and *e* edges has *e-n+2* regions.

   (b) In any simple, connected planar graph with *f* regions, *n* vertices and *e* edges (e > 2), show that the following inequalities must hold :

   (i) $e \geq \frac{3}{2} f$ and (ii) $e \geq 3n - 6$

10 (a) Show that the set consisting of all the circuits and the edge-disjoint unions of circuits (including the null set ϕ) in a graph *G* is an abelian group under the ring-sum operation $\oplus$.

   (b) Show that the set consisting of all the cut-sets and the edge-disjoint unions of cut-sets (including the null set ϕ) in a graph *G* is an abelian group under the ring-sum operation.